Two-to-One Resonances near the Equilateral Libration Points

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An analysis is performed for the nonlinear motion of a particle near the equilateral points in the restricted problem of three bodies for the case $\mu \approx \mu_2$, i.e., $\sigma = \omega_1 - 2\omega_2 \approx 0$ (ω_1 and ω_2 are the frequencies of the two modes of oscillation of the linearized equations). The analysis is carried out to second order in amplitude using both the method of multiple scales and a method of averaging. It is found that the results of both methods are in full agreement. Also, it is found that long period orbits are given by $[2(\omega_2)^{1/2}A_1 - \rho]^2 + \omega_1 A_2{}^2 = \rho^2$ where A_1 and A_2 are the amplitudes of the two modes of oscillation and $\rho = \sigma\ \sin n\pi/2/4C(\omega_2)^{1/2}$, $n = -1$ and 1, where C is a positive known function of the mass ratio of the two primaries. The results show that these long period orbits are stable only if $|A_1| \leq |\sigma|\ 12C\omega_2$. Short period orbits are found to be unstable, in the sense that if they are subjected to small disturbances the resultant motions are no longer short period orbits. At $\mu = \mu_2$, all motions near L_4 are found to be unstable. The time variation of the amplitude of an aperiodic motion is presented in terms of elliptic functions.

I. Introduction

IN the two-dimensional restricted problem of three bodies, two of the three bodies have finite masses (primaries) and the third has a negligible mass. The primaries are represented by point masses revolving in circular orbits about their common mass center uninfluenced by the third body. The motion of this third body is governed by the primaries, and remains in their plane of motion.

In a coordinate system rotating with the primaries, Lagrange[1,2] showed that there are five equilibrium points—three are colinear with the primaries (L_1, L_2, and L_3) and the other two are at the apexes of two equilateral triangles (L_4 and L_5) whose other vertices are at the primaries. A linear analysis of the motion around these points shows that the first three are unstable for all μ [$\mu = m_2/(m_1 + m_2)$, m_1 and m_2 are the masses of the two primaries]. On the other hand, the linear analysis shows that L_4 and L_5 are unstable if $\mu < \tilde{\mu} = [1 - (69)^{1/2}/9]/2$. Below $\tilde{\mu}$, the motion is stable and can be represented as the superposition of two oscillatory modes with circular frequencies ω_1 and ω_2. The value $\tilde{\mu}$ is referred to as the critical mass ratio because it separates stable from unstable triangular points.

Leontivac[3] proved that L_4 and L_5 are stable in the sense of Kolmogorov-Arnold-Moser for all $\mu < \tilde{\mu}$ except on a set of measure zero. Deprit and Deprit-Bartholomé[4] proved that the exceptional set contains at most four values of μ including $\mu_2 = 0.024295$ and $\mu_3 = 0.013516$. Markeev[5] proved instability of L_4 and L_5 at μ_2 and μ_3.

Nayfeh and Kamel[6] analyzed the nonlinear motion of a particle near L_4 and L_5 for $\mu \approx \mu_3$ using Lie transforms[7,8] and the method of multiple scales.[9,10] The results of both methods are found to be in full agreement with each other.

The purpose of the present paper is to study the nonlinear motion of a particle near L_4 and L_5 for $\mu \approx \mu_2$. The analysis is carried out using both the method of multiple scales and a method of averaging. The results of both methods are found to be also in full agreement with each other.

Subsequent to the submittal of the present paper, the author learned about two recent studies by Henrard[11] and Alfriend[12] concerning the case $\mu \approx \mu_2$. Henrard attacked this problem by normalization of the Hamiltonian in a manner somewhat analogous to the analysis of Ref. 13; while Alfriend employed the methods of multiple scales. Alfriend neither analyzes the family of long period orbits and their stability, nor the general stability of nonperiodic orbits.

II. Problem Formulation

In the restricted problem of three bodies, two of these bodies have finite masses (m_1 and m_2) and revolve around one another in circular orbits with frequency n $\{n = [G(m_1 + m_2)/D^3]^{1/2}$, G is the Newtonian gravitational constant and D is the mean distance between the finite masses$\}$. The third body has an infinitesimal mass and moves in the field of the finite masses without affecting their motion. To analyze the motion of the infinitesimal mass, a Cartesian coordinate system rotating with the finite masses and centered at their center of mass is introduced such that they lie along the $\bar{x}$ axis. If distances and time are made dimensionless using the characteristic length D and characteristic frequency n, the equations of motion of the infinitesimal mass are[2]

$$\ddot{\bar{x}} - 2\dot{\bar{y}} = -\partial V/\partial \bar{x} \tag{2.1a}$$

$$\ddot{\bar{y}} + 2\dot{\bar{x}} = \partial V/\partial \bar{y} \tag{2.1b}$$

$$-V = (\bar{x}^2 + \bar{y}^2)/2 + (1 - \mu)/r_1 + \mu/r_2 \tag{2.1c}$$

$$r_1{}^2 = (\bar{x} - \bar{x}_1)^2 + \bar{y}^2,\ r_2{}^2 = (\bar{x} - \bar{x}_2)^2 + \bar{y}^2 \tag{2.1d}$$

where dots denote differentiation with respect to time, $\bar{x}_1 = -\mu$ and $\bar{x}_2 = 1 - \mu$. There are five positions of equilibrium for the infinitesimal mass. They are given by the equation[1,2]

$$\nabla V = 0 \tag{2.2}$$

Three of these equilibrium points (L_1, L_2, and L_3) lie along the $\bar{x}$ axis and are unstable. The other two points (L_4 and L_4) form equilateral triangles in the $\bar{x}$-$\bar{y}$ plane with the two finite masses.

To investigate the stability of L_4 (the same results hold for L_5), the body is displaced to

$$\bar{x} = (1 - 2\mu)/2 + x,\ \bar{y} = (3)^{1/2}/2 + y \tag{2.3}$$

The right-hand sides of Eqs. (2.1a) and (2.1b) are expanded in Taylor series keeping up to second-order terms in x and y. The resulting equations are

$$\ddot{x} - 2\dot{y} - 3x/4 - \eta y = -H_x{}^{(1)} \tag{2.4a}$$

Received December 22, 1969; presented as Paper 70-98 at the AIAA 8th Aerospace Sciences Meeting, New York, January 19–21, 1970; revision received May 11, 1970. It is a pleasure to acknowledge the many useful discussions with A. A. Kamel, and the comments of A. Deprit and K. T. Alfriend. The author thanks very much R. M. Kendall for his presentation of this paper at the AIAA meeting.

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$$\ddot{y} + 2\dot{x} - \eta x - 9y/4 = -H_y^{(1)} \quad (2.4b)$$

$$H^{(1)} = 3(3)^{1/2}y(x^2 + y^2)/16 + (3)^{1/2}\eta x(33y^2 - 7x^2)/16 \quad (2.4c)$$

where $\eta = 3(3)^{1/2}(1 - 2\mu)/4$, and the subscripts x and y denote partial derivatives. These equations are appropriate for the application of the method of multiple scales, but the corresponding Hamiltonian is more appropriate for applications of the method of averaging. If the conjugate momenta p_x and p_y are defined by

$$p_x = \dot{x} - y,\ p_y = \dot{y} + x \quad (2.5a)$$

then the Hamiltonian corresponding to Eq. (2.4) is

$$H = H^{(0)} + H^{(1)} \quad (2.5b)$$

where

$$H^{(0)} = (p_x^2 + p_y^2)/2 + (yp_x - xp_y) + (x^2 - 5y^2)/8 - \eta xy \quad (2.5c)$$

The equations of motion Eq. (2.4) can be written in terms of H as

$$\dot{x} = H_{p_x},\ \dot{p}_x = -H_x \quad (2.6a)$$

$$\dot{y} = H_{p_y},\ \dot{p}_y = -H_y \quad (2.6b)$$

If the nonlinear terms $H^{(1)}$ are neglected, the solution of the resulting equations (corresponding to $H^{(0)}$) can be obtained by assuming that x and y are proportional to $\exp(i\omega t)$ where

$$\omega^4 - \omega^2 + (\tfrac{27}{16}) - \eta^2 = 0 \quad (2.7)$$

The four roots of this equation depend on the value of μ. If $0 \le \mu < \tilde{\mu} = \frac{1}{2}[1 - (69)^{1/2}/9]$, the four roots ($\pm\omega_1$ and $\pm\omega_2$) are real, and hence the equilibrium of a particle at L_4 is stable. On the other hand, if $\mu \ge \tilde{\mu}$, the equilibrium is unstable. In the stable case the solution can be written in the form[13,14]

$$\begin{bmatrix} x \\ y \\ p_x \\ p_y \end{bmatrix} = [J] \begin{bmatrix} (A_1/\omega_1)\sin B_1 \\ (A_2/\omega_2)\sin B_2 \\ A_1 \cos B_1 \\ -A_2 \cos B_2 \end{bmatrix} \quad (2.8a)$$

where

$$A_i = (2\alpha_i)^{1/2},\ B_1 = \omega_1(t + \beta_1),\ B_2 = \omega_2(t - \beta_2) \quad (2.8b)$$

If we let

$$k_i = |11\omega_i^2/2 + 2\eta^2 - 45/8|^{-1/2} \quad (2.8c)$$

Then,

$$[J] = \begin{bmatrix} 0 & 0 & (k_1/\omega_1)(\omega_1^2 + \frac{9}{4}) & -[k_2/\omega_2](\omega_2^2 + \frac{9}{4}) \\ -2k_1\omega_1 & -2k_2\omega_2 & -(k_1/\omega_1)\eta & (k_2/\omega_2)\eta \\ -k_1\omega_1(\omega_1^2 + \frac{1}{4}) & -k_2\omega_2(\omega_2^2 + \frac{1}{4}) & (k_1/\omega_1)\eta & -(k_2/\omega_2)\eta \\ k_1\omega_1\eta & k_1\omega_2\eta & (k_1/\omega_1)(\frac{9}{4} - \omega_1^2) & -[k_2/\omega_2](\frac{9}{4} - \omega_2^2) \end{bmatrix}$$

Here, β_1, β_2, α_1, and α_2 are the constants of integration. Subsitution of (2.8a) into $H^{(0)}$ leads to

$$H^{(0)} = \alpha_1 - \alpha_2 \quad (2.9)$$

Hence, B_1/ω_1, $-B_2/\omega_2$ are the coordinates and the α_i's are the conjugate momenta of a canonical set. Moreover, the quantities α_i and β_i form a canonical set with respect to a transformed Hamiltonian $H^{(0)} = 0$.

Since nonlinear terms are neglected, the previous solution is valid only for very small amplitudes A_1 and A_2. Any attempt to extend the aforementioned results to larger amplitudes by a straightforward perturbation or by iteration will fail whenever two integers n and m can be found such that $n\omega_1 + m\omega_2$ is zero or near zero due to the appearance of secular terms or small divisors. In these cases, alternative methods such as the method of multiple scales[9,10] and Hamiltonian methods with suitable canonical transformations[7,8,15] should be used. In Secs. 3 and 4, respectively, the methods of multiple scales and averaging are employed to study the stability of L_4 and L_5 for small $\sigma = \omega_1 - 2\omega_2$, i.e., $\mu \approx \mu_2$.

III. Method of Multiple Scales

A perturbation solution for finite but small amplitudes is obtained in this section using the method of multiple scales. According to this method, x and y are assumed to be functions of two independent time scales—a fast time $T_0 = t$, and a slow time $T_1 = \epsilon t$, where ϵ is a small parameter of the order of the amplitude of the first-order solution. To second-order terms in ϵ, x and y are assumed to possess the following uniformly valid expansions for all times

$$x(t;\epsilon) = \epsilon x_1(T_0,T_1) + \epsilon^2 x_2(T_0,T_1) + 0(\epsilon^3) \quad (3.1)$$

$$y(t;\epsilon) = y_1(T_0,T_1) + \epsilon^2 y_2(T_0,T_1) + 0(\epsilon^3) \quad (3.2)$$

The time derivative becomes

$$d/dt = D_0 + \epsilon D_1 + \dots,\ D_n = \partial/\partial T_n \quad (3.3)$$

Substituting Eqs. (3.1–3.3) into Eqs. (2.4), expanding, and equating coefficients of equal powers of ϵ on both sides lead to:
Order ϵ

$$M_1(x_1,y_1) = D_0^2 x_1 - 2D_0 y_1 - 3x_1/4 - \eta y_1 = 0 \quad (3.4)$$

$$M_2(x_1,y_1) = D_0^2 y_1 + 2D_0 x_1 - \eta x_1 - 9y/4 = 0 \quad (3.5)$$

Order ϵ^2

$$M_1(x_2,y_2) = 7(3)^{1/2}\eta x_1^2/12 - 3(3)^{1/2}x_1y_1/8 - 11(3)^{1/2}\eta y_1^2/12 - 2D_0D_1x_1 + 2D_1y_1 \quad (3.6)$$

$$M_2(x_2,y_2) = -3(3)^{1/2}x_1^2/16 - 11(3)^{1/2}\eta x_1y_1/6 - 9(3)^{1/2}y_1^2/16 - 2D_0D_1y_1 - 2D_1x_1 \quad (3.7)$$

where the operators M_1 and M_2 are defined by Eqs. (3.4) and (3.5).

The general solution of Eqs. (3.4) and (3.5) is given by (2.8) if A_i and β_i are considered functions of the slow time T_1 rather than being constants. These arbitrary functions will be determined in the course of analysis. If A_i is small then ϵ can be used only to keep track of the ordering, and will be equated to one in the final solution.

Substitution of Eqs. (2.8) into the right-hand sides of Eqs. (3.6) and (3.7) gives

$$M_1(x_2,y_2) = \sum_{i=1}^{2} (P_{1i}\sin B_i + Q_{1i}\cos B_i) + \text{nonsecular producing terms} \quad (3.8a)$$

$$M_2(x_2,y_2) = \sum_{i=1}^{2} (P_{21}\sin B_1 + Q_{2i}\cos B_i) + \text{nonsecular producing terms} \quad (3.8b)$$

where the P's and Q's are given in Appendix A. They are functions of $\tilde{\gamma}$ ($\tilde{\gamma} = \sigma T_0 + \omega_1\beta_1 + 2\omega_2\beta_2$) only, which can be considered a function of T_1 if the detuning σ ($\sigma = \omega_1 - 2\omega_2$) is small. In this case, the particular solution of Eqs. (3.8a) and (3.8b) contains secular terms with respect to the time scale T_0 which make x_2 and y_2 unbounded as $T_0 \to \infty$; that is as $t \to \infty$. Hence x_2 and y_2 will dominate x_1 and y_1, and the expansion will break down for large t unless the secular terms are eliminated.

One can determine the conditions which must be satisfied for there to be no secular terms by obtaining the particular solutions and then setting the coefficients of the secular terms equal to zero. A simpler method to determine these conditions proceeds as follows. Let

$$x_2 = 0,\ y_2 = a_i \cos B_i + b_i \sin B_i \tag{3.9}$$

Substitution of these expressions into Eqs. (3.8a) and (3.8b), and equating the coefficients of $\cos B_i$ and $\sin B_i$ on both sides lead to

$$2\omega_i a_i - \eta b_i = P_{1i} \tag{3.10a}$$

$$-2\omega_i . b_i - \eta a_i = Q_{1i} \tag{3.10b}$$

$$-(\omega_i^2 + \tfrac{9}{4})a_i = Q_{2i} \tag{3.10c}$$

$$-(\omega_i^2 + \tfrac{9}{4})b_i = P_{2i} \tag{3.10d}$$

Elimination of a_i and b_i from the above four equations yields the required conditions as

$$-2\omega_i Q_{2i} + \eta P_{2i} = (\omega_i^2 + \tfrac{9}{4})P_{1i} \tag{3.11a}$$

$$2\omega_i P_{2i} + \eta Q_{2i} = (\omega_i^2 + \tfrac{9}{4})Q_{1i} \tag{3.11b}$$

$i = 1$ and 2.

Substituting for the P's and Q's from Appendix A and solving the resulting equations for A_i' and B_i' lead to

$$A_1' = \omega_1 C A_2^2 \cos\gamma \tag{3.12}$$

$$A_2' = 2\omega_2 C A_1 A_2 \cos\gamma \tag{3.13}$$

$$A_1\beta_1' = -CA_2^2 \sin\gamma \tag{3.14}$$

$$\beta_2' = -2CA_1 \sin\gamma \tag{3.15}$$

Primes denote differentiation with respect to T_1, and C and γ are given in Appendix B. Equations (3.12–3.16) can be shown to correspond to Eqs. (32a–32d) of Alfriend[12] by changing his variables to ours or vice versa. Since $\gamma' = \sigma + \omega_1\beta_1' + 2\omega_2\beta_2'$, Eqs. (3.14) and (3.15) can be combined into

$$A_1\gamma' = A_1\sigma - C(\omega_1 A_2^2 + 4\omega_2 A_1^2)\sin\gamma \tag{3.16}$$

Integrals of Motion

Elimination of γ from Eqs. (3.12) and (3.13), and integration of the resulting equation yield

$$2\omega_2 A_1^2 = \omega_1 A_2^2 + \nu_1 \tag{3.17}$$

where ν_1 is a constant of integration. To determine a second integral, let

$$z = C\omega_1 \sin\gamma \tag{3.18}$$

Elimination of γ and t from Eqs. (3.12, 3.16 and 3.18) gives

$$A_1A_2^2(dz/dA_1) + (A_2^2 + 4\omega_2 A_1/\omega_1)z = A_1\sigma \tag{3.19}$$

Since $2\omega_2 A_1 dA_1 = \omega_1 A_2 dA_2$ from Eq. (3.17), the solution of Eq. (3.19) is

$$A_1^2\sigma/2 - zA_1A_2^2 = \nu_2 \tag{3.20}$$

where ν_2 is another constant of integration.

Periodic Orbits and Their Stability

Before considering long period orbits (L_4^l), let us consider short period orbits (L_4^s). Equations (3.12–3.15) admit stationary solutions corresponding to L_4^s (i.e., $A_2 = 0$ and $A_1 = a$ = a const) with a constant period independent of the amplitude a to second order (i.e., $\tau = 2\pi/\omega_1$). However, Eqs. (3.12–3.15) show that any small disturbance applied to an L_4^s orbit results in an aperiodic, though stable, orbit. Thus, in this sense, the family L_4^s is unstable.

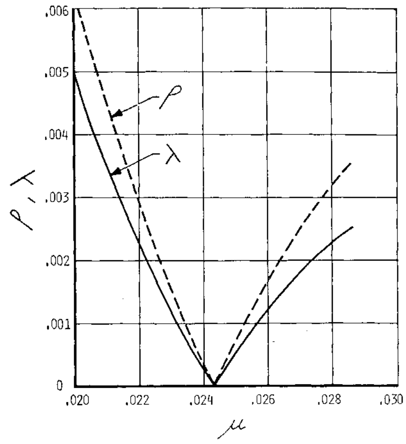
Long period orbits to second order correspond to the stationary solutions of Eqs. (3.12, 3.13, and 3.16). They are given by

$$\cos\gamma_0 = 0 \text{ or } \gamma_0 = n\pi/2 \tag{3.21}$$

with n an odd integer, and

$$A_{10}\sigma - C(\omega_1 A_{20}^2 + 4\omega_2 A_{10}^2)\sin\gamma_0 = 0 \tag{3.22}$$

Fig. 1 Variation of ρ and λ with μ.



Introducing the variables $\tilde{A}_{20} = (\omega_1)^{1/2}A_{20}$ and $\tilde{A}_{10} = (\omega_2)^{1/2}A_{10}$ transforms Eq. (3.22) into

$$(\tilde{A}_{10} - \rho)^2 + \tilde{A}_{20}^2 = \rho^2 \tag{3.23a}$$

where

$$\rho = \sigma \sin\gamma_0/4C(\omega_2)^{1/2} \tag{3.23b}$$

Thus, long period orbits are given by a circle whose center is $(\rho,0)$ and radius ρ in the $\tilde{A}_{10} - \tilde{A}_{20}$ plane. The variation of ρ with μ is shown in Fig. 1. For $\sigma = 0$ ($\omega_1 - 2\omega_2$), $\rho = 0$, and consequently there are no finite amplitude long period orbits to second order.

The general long period orbits are described by a superposition of both normal modes (i.e., $A_{10} \neq 0$ and $A_{20} \neq 0$). Periodicity is achieved through adjustment of the frequencies of both modes via the nonlinear coupling which makes them exactly commensurable. The frequencies of the two modes are given by

$$\tilde{\omega}_1 = \omega_1[1 - CA_{20}^2/A_{10}] = 2\omega_2[1 + 2CA_{10}] = 2\tilde{\omega}_2 \tag{3.24}$$

To determine the stability of these orbits, let

$$\tilde{A}_1 = \tilde{A}_{10} + \Delta A_1 \exp(sT_1) \tag{3.25}$$

$$A_2 = A_{20} + A_2 \exp(sT_1) \tag{3.26}$$

$$\gamma = \gamma_0 + \Delta\gamma \exp(sT_1) \tag{3.27}$$

Substituting Eqs. (3.25–3.27) into Eqs. (3.12, 3.13, and 3.16), and using Eqs. (3.21) and (3.22) give

$$s\Delta A_1 = -2(\omega_2)^{1/2}C\tilde{A}_{20}^2 \sin\gamma_0\Delta\gamma \tag{3.28}$$

$$s\Delta A_2 = -(\omega_2)^{1/2}C\tilde{A}_{10}\tilde{A}_{20}\sin\gamma_0\Delta\gamma \tag{3.29}$$

$$\tilde{A}_{10}s\Delta\gamma = [\sigma - 4C(\omega_2)^{1/2}\tilde{A}_{10}\sin\gamma_0]\Delta A_1 - 4C(\omega_2)^{1/2}\tilde{A}_{20}\sin\gamma_0\Delta A_2 \tag{3.30}$$

Elimination of ΔA_1, ΔA_2, and $\Delta\gamma$ from Eqs. (3.28–3.20) yields

$$s^2 = 12C^2\omega_2(\tilde{A}_{10} - 2\rho/3)\tilde{A}_{20}^2/\tilde{A}_{10} \tag{3.31}$$

Thus, a periodic orbit is stable if

$$|\tilde{A}_{10}| \leq 2|\rho|/3 \tag{3.32a}$$

or

$$|A_{10}| \leq 2\lambda/3 = \sigma\sin\gamma_0/12C\omega_2 \tag{3.32b}$$

Otherwise, it is unstable. The variation of λ with μ is also shown in Fig. 1.

Stability of Nonperiodic Motions

Let

$$\xi = \omega_1 A_2^2/|\nu_1| \tag{3.33a}$$

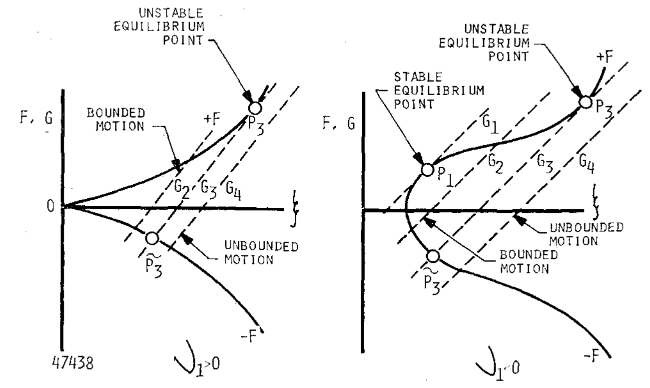


Fig. 2 Topology of long period orbits.

Then, from Eq. (3.17),

$$A_1^2 = |\nu_1|(\xi \pm 1)/2\omega_2 \tag{3.33b}$$

Eliminating γ from Eqs. (3.13, 3.18, and 3.20), using Eqs. (3.33), and rearranging give

$$\xi'^2 = \nu_3^2[F^2(\xi) - G^2(\xi)] \tag{3.34a}$$

$$F = \pm\xi(\xi \pm 1)^{1/2} \tag{3.34b}$$

$$G(\xi) = \nu_4(\xi + \tilde{\nu}_2) \tag{3.34c}$$

$$\nu_3^2 = 8\omega_2 C^2|\nu_1|, \quad \nu_4^2 = \sigma^2/8\omega_2 C^2|\nu_1|,$$
$$\tilde{\nu}_2 = 4\nu_2\omega_2/\sigma|\nu_1| \pm 1 \tag{3.34d}$$

where the plus and minus signs in the parenthesis correspond to positive and negative values of ν_1, respectively.

The functions $F(\xi)$ and $G(\xi)$ are shown schematically in Fig. 2. Since A_2, and hence ξ, must be real, $F^2 \geq G^2$ must hold. The points where G meets F correspond to the vanishing of both A_1' and A_2'. A curve such as G_2 which meets both branches of F, or meets one branch at two different points, corresponds to a bounded aperiodic motion. Let the two points of intersection be denoted by ξ_1 and ξ_2 where $\xi_2 > \xi_1$. Since $F^2 - G^2$ is cubic in ξ, there is a third real root ξ_3. Investigation of Fig. 2 shows that $\xi_3 > \xi_2$. Hence,

$$t - t_0 = \nu_3 \int_{\xi_1}^{\xi} [(\tau - \xi_1)\cdot(\xi_2 - \tau)(\xi_3 - \tau)]^{-1/2} d\tau \tag{3.35a}$$

where $\xi(t_0) = \xi_1$. Thus

$$t - t_0 = 2\nu_3(\xi_3 - \xi_1)^{-1/2} \operatorname{sn}^{-1}[(\xi - \xi_1)^{1/2}(\xi_2 - \xi_1)^{-1/2},$$
$$(\xi_2 - \xi_1)^{1/2}(\xi_3 - \xi_1)^{-1/2}] \tag{3.35b}$$

On the other hand, a curve such as G_4 which meets F at one point only represents unbounded motion. However, the points P_1 and P_3 where G_1 and G_3 touch F (i.e., $\xi_2 = \xi_1$ or ξ_3) represent equilibrium (periodic orbits) because the relationship $F' = G'$ is equivalent to Eqs. (3.21) and (3.22). A point such as P_1 corresponds to a stable periodic orbit, whereas a point such as P_3 corresponds to an unstable periodic orbit.

At $\mu = \mu_2$, $\sigma = 0$, and hence, $G(\xi) =$ constant. Thus, Fig. 2 shows that, in this case, G intersects F in one point only. Consequently, to second order, all motions near L_4 and L_5 are unstable for $\mu = \mu_2$, in agreement with the result of Markeev.[5]

IV. Method of Averaging

The solution of the nonlinear problem corresponding to the Hamiltonian (2.5b) can be obtained starting from the linear solution using the method of variation of parameters. If α_i and β_i are allowed to be functions of time rather than constants, then the form of the solution remains the same as the linear solution (2.8). Substituting the linear solution into $H^{(1)}$ gives

$$H^{(1)} = -2(2)^{1/2}C\alpha_1\alpha_2 \sin\gamma \tag{4.1}$$

plus short period terms where C and γ are given in Appendix B. Then, α_i and β_i satisfy Hamilton's equations corresponding to $H^{(1)}$; namely,

$$\dot{\alpha}_i = -\partial H^{(1)}/\partial\beta_i, \; \dot{\beta}_i = \partial H^{(1)}/\partial\alpha_i \tag{4.2}$$

In order to determine an approximate second-order solution for the motion of a particle near the equilateral libration points, the long period behavior (slow time variation) only needs to be investigated. Thus, we average Eqs. (4.2) over $T = 2\pi/\omega_2$ which is equivalent to neglecting the short period terms in Eqs. (4.1) and (4.2). Thus,

$$\tilde{H} = -2(2)^{1/2}C\alpha_1\alpha_2 \sin\gamma \tag{4.3}$$

where $\tilde{H}$ is the average of $H^{(1)}$. Hence, the equations for α_i and β_i become

$$\dot{\alpha}_1 = 2(2)^{1/2}C\omega_1\alpha_1^{1/2}\alpha_2 \cos\gamma \tag{4.4}$$

$$\dot{\alpha}_2 = 4(2)^{1/2}C\omega_2\alpha_1^{1/2}\alpha_2 \cos\gamma \tag{4.5}$$

$$\dot{\beta}_1 = -(2)^{1/2}C\alpha_1^{-1/2}\alpha_2 \sin\gamma \tag{4.6}$$

$$\dot{\beta}_2 = -2(2)^{1/2}C\alpha_1^{1/2} \sin\gamma \tag{4.7}$$

Since $\alpha_i = A_i^2/2$, Eqs. (4.4–4.7) are identical to Eqs. (3.12–3.15). Thus, the results obtained by the method of averaging are in full agreement with those obtained by the method of multiple scales.

The constant of integration ν_2 of Eq. (3.20) can be related to the averaged Hamiltonian. To determine the relationship, the explicit time dependence of H is removed first. A new canonical set α_i^*, β_i^* is introduced via the generating function S:

$$S = \alpha_1^*(\sigma t/\omega_1 + \beta_1) + \alpha_2^*\beta_2 \tag{4.8}$$

Thus,

$$\alpha_1 = \partial S/\partial\beta_1 = \alpha_1^* \tag{4.9}$$

$$\alpha_2 = \partial S/\partial\beta_2 = \alpha_2^* \tag{4.10}$$

$$\beta_1^* = \partial S/\partial\alpha_1^* = \sigma t/\omega_1 + \beta_1 \tag{4.11}$$

$$\beta_2^* = \partial S/\partial\alpha_2^* = \beta_2 \tag{4.12}$$

Then, the transformed Hamiltonian K can be obtained from

$$K = \tilde{H}(\alpha^*,\beta^*) + \partial S(\alpha^*,\beta^*,t)/\partial t \tag{4.13}$$

and is given by

$$K = -2(2)^{1/2}C\alpha_1^{*1/2}\alpha_2^* \sin\gamma + \sigma\alpha_1^*/\omega_1 \tag{4.14}$$

with $\gamma = \omega_1\beta_1^* + 2\omega_2\beta_2^* + \phi$. Since $\alpha_i^* = \alpha_i = A_i^2/2$, comparison of Eqs. (4.14) and (3.20) gives

$$K = \nu_2/\omega_1 \tag{4.15}$$

V. Summary

Second-order expansions are presented for the motion of a particle near the equilateral points in the restricted problem of three bodies for values of μ near $\mu_2 = 0.024295$. The analysis is carried out using both a method of averaging and the method of multiple scales.[9,10] It is found that the results obtained by the aforementioned two methods are in full agreement with each other.

For the previous range of μ the linearized analysis[1,2] predicts stable solutions having two modes of oscillation with amplitude independent frequencies ω_1 and ω_2 ($\sigma = \omega_1 - 2\omega_2 \approx 0$). Moreover, it predicts that stable periodic orbits with either frequency are possible. However, the nonlinear theory shows that periodic orbits have in general amplitude dependent frequencies. It predicts that short period orbits are unstable, while long period orbits are stable for all A_1 such that $|A_1| < |\sigma|/12C\omega_2$ where C is a known positive function of μ. In the special case $\sigma = 0$ (i.e., $\mu = \mu_2$), there are no finite amplitude long period orbits according to the present second-order nonlinear theory.

If both A_1 and A_2 are different from zero, then long period orbits are given by

$$[2(\omega_2)^{1/2}A_1 - \rho]^2 + \omega_1 A_2^2 = \rho^2 \quad (5.1)$$

where

$$\rho = \sigma \sin n\pi/2/4C(\omega_2)^{1/2},\ n = 1 \text{ and } -1 \quad (5.2)$$

It is found that these periodic orbits are stable only if

$$|A_1| \leq |\sigma|/12C\omega_2 \quad (5.3)$$

If μ is such that $\sigma \neq 0$, Eq. (5.1) shows that for each $|A_1| \leq |\rho|/(\omega_2)^{1/2}$ there corresponds only one value of $|A_2|$, while for each $|A_2| \leq |\rho|/(\omega_2)^{1/2}$ there corresponds one or two values of $|A_1|$.

The stability of nonperiodic motions is also investigated. It is found that $\xi(\propto A_2^2)$ is given by

$$\xi'^2 = h(\xi) \quad (5.4)$$

where h is a third-order polynomial in ξ. If h has three real roots such that $\xi_1 < \xi_2 < \xi_3$, then the motion is aperiodic and ξ oscillates between ξ_1 and ξ_2. The special case $\xi_2 = \xi_3$ corresponds to unstable periodic orbits, whereas the case $\xi_1 = \xi_2$ corresponds to stable periodic orbits. On the other hand, if h has only one real root, then the motion is unstable.

It should be mentioned that our results are not valid outside the interval $\mu^* \leq \mu < \tilde{\mu}$. Below μ^*, long period orbits terminate on a short period orbit travelled three times, rather than two times as we assumed. Henrard[16] has shown that $0.02070 < \mu^* < 0.02075$. Above $\tilde{\mu}$, the motion is unstable according to the linear theory as discussed in the introduction.

Appendix A

$$P_{1i} = (-1)^{i+1}e_{1i}\sin\tilde{\gamma} + d_{1i}\cos\tilde{\gamma} + 2(\omega_i a_{1i} + b_{2i})A_i' - 2\omega_i a_{2i}A_i\beta_i' \quad (A1)$$

$$P_{2i} = (-1)^{i+1}e_{2i}\sin\tilde{\gamma} + d_{2i}\cos\tilde{\gamma} + 2\omega_i a_{2i}A_i' + 2\omega_i(\omega_i b_{2i} + a_{1i})A_i\beta_i' \quad (A2)$$

$$Q_{1i} = e_{1i}\cos\tilde{\gamma} + (-1)^i d_{1i}\sin\tilde{\gamma} + 2a_{2i}A_i' + 2\omega_i(\omega_i a_{1i} + b_{2i})A_i\beta_i' \quad (A3)$$

$$Q_{2i} = e_{2i}\cos\tilde{\gamma} + (-1)^i d_{2i}\sin\tilde{\gamma} - 2(a_{1i} + \omega_i b_{2i})A_i' + 2\omega_i^2 a_{2i}A_i\beta_i' \quad (A4)$$

$$e_{11} = [21ua_{12}^2 - 6(3)^{1/2}a_{12}a_{22} - 33ua_{22}^2 + 33b_{22}^2]A_2^2/32 \quad (A5)$$

$$e_{12} = [21ua_{11}a_{12} - 3(3)^{1/2}(a_{21}a_{11} + a_{22}a_{11}) - 33u(a_{21}a_{22} + b_{21}b_{22})]A_1A_2/16 \quad (A6)$$

$$e_{21} = [-3(3)^{1/2}a_{12}^2 - 66ua_{12}a_{22} - 9(3)^{1/2}a_{22}^2 + 9(3)^{1/2}b_{22}^2]A_2^2/32 \quad (A7)$$

$$e_{22} = [-3(3)^{1/2}a_{11}a_{12} - 33u(a_{21}a_{11} + a_{22}a_{11}) - 9(3)^{1/2}(a_{21}a_{22} + b_{21}b_{22})]A_1A_2/16 \quad (A8)$$

$$d_{11} = -[3(3)^{1/2}a_{12}b_{22} + 33ua_{22}b_{22}]A_2^2/16 \quad (A9)$$

$$d_{12} = -[3(3)^{1/2}(a_{12}b_{21} - a_{11}b_{22}) + 33u(a_{22}b_{21} - a_{21}b_{22})]A_1A_2/16 \quad (A10)$$

$$d_{21} = -[33ua_{12}b_{22} + 9(3)^{1/2}a_{22}b_{22}]A_2^2/16 \quad (A11)$$

$$d_{22} = -[33u(a_{12}b_{21} - a_{11}b_{22}) + 18(3)^{1/2}(a_{22}b_{21} - a_{21}b_{22})]A_1A_2/32 \quad (A12)$$

$$u = 1 - 2\mu \quad (A13)$$

$$a_{1i} = k_i(\omega_i^2 + 9/4)/\omega_i \quad (A14)$$

$$a_{2i} = -k_i\eta/\omega_i \quad (A15)$$

$$b_{2i} = -2k_i \quad (A16)$$

Appendix B

$$C = (C_1^2 + C_2^2)^{1/2} \quad (B1)$$

$$\gamma = \tilde{\gamma} + \phi,\ \tan\phi = C_1/C_2 \quad (B2)$$

$$64C_1 = 33ub_{22}^2a_{11} - 66ua_{12}b_{21}b_{22} - 33ua_{11}a_{22}^2 - 66ua_{12}a_{21}a_{22} + 21ua_{11}a_{12}^2 + 9(3)^{1/2}a_{21}b_{22}^2 - 18(3)^{1/2}a_{22}b_{21}b_{22} - 3(3)^{1/2}a_{21}a_{22}^2 - 6(3)^{1/2}a_{11}a_{12}a_{22} - 3(3)^{1/2}a_{12}^2a_{21} \quad (B3)$$

$$64C_2 = 66ua_{11}a_{22}b_{22} + 66ua_{12}a_{21}b_{22} - 66ua_{12}a_{22}b_{21} + 9(3)^{1/2}b_{21}b_{22} - 18(3)^{1/2}a_{21}a_{22}b_{21} - 6(3)^{1/2}a_{11}a_{12}b_{22} - 9(3)^{1/2}a_{22}^2b_{21} - 3\ (3)^{1/2}a_{12}^2b_{21} \quad (B4)$$

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